Piecewise linear bounding of univariate nonlinear functions and resulting MILP-based solution methods

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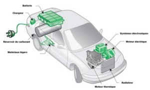
Outlook

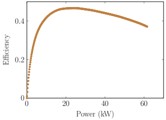
Energy in hybrid electric vehicles

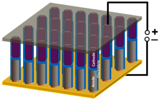


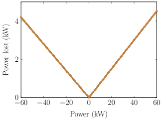
Electric propulsion motor powered by :

- onboard generator : e.g. hydrogen fuel cell (FC)
- reversible source : e.g. supercapacitator (SE)

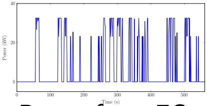
Energy sources characteristics : power limits(kW), efficiency(%), capacity(kWs) ...





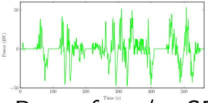


Find at each instant the optimal power split between the energy sources to minimize the total fuel consumption.



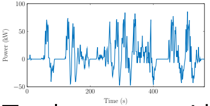
Power from FC

+



Power from/to SE

=



Total power provided

Mathematical model

$$\min \sum_{i=1}^{n} f^{\mathrm{FC}}(x_i) \tag{1}$$

s.t. Power demand satisfaction

$$x_i + y_i - z_i \geq P_i, \qquad \forall i \in \mathbb{I} \tag{2}$$

Final SE energy level higher or equal to initial energy level

$$\sum_{i=1}^{n} f^{\mathrm{SE}^+}(y_i) - f^{\mathrm{SE}^-}(z_i) \leq 0 \tag{3}$$

SE energy level within bounds

$$E_0^{\mathrm{SE}} - E_{\max}^{\mathrm{SE}} \leq \sum_{k=1}^{i} f^{\mathrm{SE}^+}(y_k) - f^{\mathrm{SE}^-}(z_k) \leq E_0^{\mathrm{SE}} - E_{\min}^{\mathrm{SE}}, \qquad \forall i \in \mathbb{I} \tag{4}$$

Variables domains

$$x_i \in \{0\} \cup [P_{\min}^{\mathrm{FC}}, P_{\max}^{\mathrm{FC}}], y_i \in [0, P_{\max}^{\mathrm{SE}}], z_i \in [0, P_{\min}^{\mathrm{SE}}], \qquad \forall i \in \mathbb{I} \tag{5}$$

Water pumping and desalination process

Electrical model

- V_m, I_m : electrical tension, current
- T_m : motor electromag. torque
- Ω : rotation speed
- k_Φ : torque equivalent coefficient
- r : stator resistance

Electric motor equations (inertia neglected) :

$$V_m = rI_m + k_\Phi \Omega \quad (6)$$

$$T_m = \Phi_m I_m \quad (7)$$

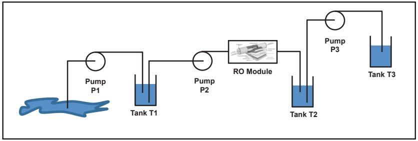
Electrical power needed : $P_e = V_m I_m$.

Pressure drop in the pipe

- $\Delta \mathrm{Pipe}, \rho$: pressure drop,water density
- h : height of water pumping

Static+Dynamic pressure

$$\Delta \mathrm{Pipe} = kq^2 + \rho g h \quad (8)$$



Mechanical-Hydraulic conv.

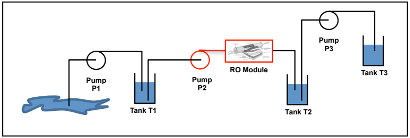
- P_p : output pressure
- q : debit of water
- a, b : non linear girator coefs
- c : hydraulic friction
- p_0 : suction pressure
- $s_p + s_m$: mechanical losses

Static equations of the motor-pump (mechanical inertia neglected) :

$$P_p = (a\Omega + bq)\Omega - (cq^2 + p_0) \quad (9)$$

$$T_m = (a\Omega + bq)q + (s_m + s_p)\Omega \quad (10)$$

Efficiency function of pump 2 + RO module



Subsystem pump 2 + Reverse Osmosis module is modeled with

$$\mathbf{P_e} = r*\mathcal{K}(\mathbf{q},\mathbf{h})+((s_m+s_p)*\Omega(\mathbf{q},\mathbf{h})+(\mathbf{q}+\mathcal{F}(\mathbf{q})/R_{\mathrm{Me}})*\mathcal{M}(\mathbf{q},\mathbf{h}))*\Omega(\mathbf{q},\mathbf{h})$$

where

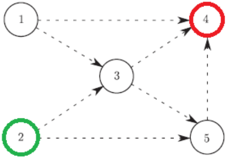
$$\begin{cases} \mathcal{F}(\mathbf{q}) = (R_{\mathrm{Mod}} + R_{\mathrm{Valve}}) * \mathbf{q}^2 \\ \mathcal{G}(\mathbf{q}) = (b * (\mathbf{q} + \mathcal{F}(\mathbf{q})/R_{\mathrm{Me}})) \\ \mathcal{M}(\mathbf{q},\mathbf{h}) = a * \Omega(\mathbf{q},\mathbf{h}) + \mathcal{G}(\mathbf{q}) \\ \Omega(\mathbf{q},\mathbf{h}) = \frac{-\mathcal{G}(\mathbf{q})+\sqrt{\mathcal{G}(\mathbf{q})^2-4a*(-(p_0+\rho g*(\mathbf{h}-l_{\mathrm{out}})+(k+c)*((\mathbf{q}+\mathcal{F}(\mathbf{q})/R_{\mathrm{Me}})^2)+\mathcal{F}(\mathbf{q})))}}{2*a} \\ \mathcal{K}(\mathbf{q},\mathbf{h}) = (((s_m + s_p) * \Omega(\mathbf{q},\mathbf{h}) + (\mathbf{q} + \mathcal{F}(\mathbf{q})/R_{\mathrm{Me}}) * (a * \Omega(\mathbf{q},\mathbf{h}) + \mathcal{G}(\mathbf{q})))/k_\phi)^2 \end{cases}$$

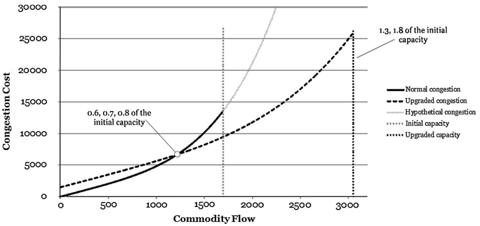
Multicommodity network design problem with congestion

Minimize Total cost = design cost + routing cost + capacity augmentation cost + congestion cost

s.t.

- Flow conservation
- Maximum capacity (with/without upgrade)
- all commodities get to destination





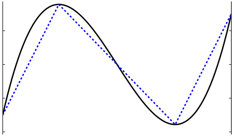
Paraskevopoulos, Sinan Gürel and Bektas, 2016

Classical MINLP solution methods

MILP-based solution methods on similar problems

Camponogara *et al.* 2011 ; Borghetti *et al.*, 2008

- approximate with piecewise linear functions



+ (more) tractable problems
- try and error approach : No guarantees on the solution quality or iterative process with an undefined number of iterations
- global optimality cannot be guaranteed

Generic MINLP solution methods / Hybrid algorithms and frameworks

Grossmann 2002 / Bonami *et al.*, 2008

+ global optimality guaranteed if carried out to completion
- only for small/medium instances

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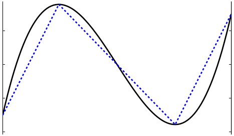

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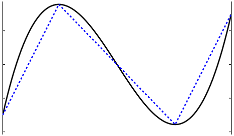

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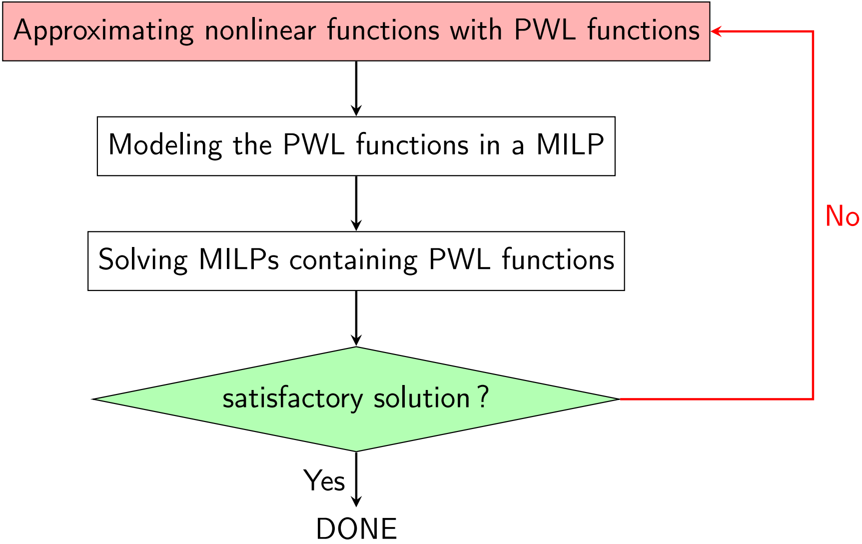
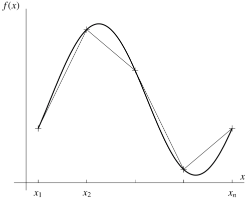
MILP-based solution methods for MINLPs



Approximating nonlinear functions with PWL functions

Sampling

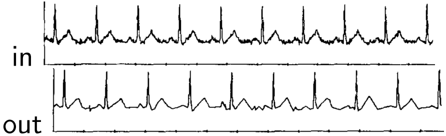


Given a number of breakpoints on the curve, find the PWL functions which minimizes an error metric.

... D'Ambrosio et al 2010 ... D'Ambrosio et al 2015 ...

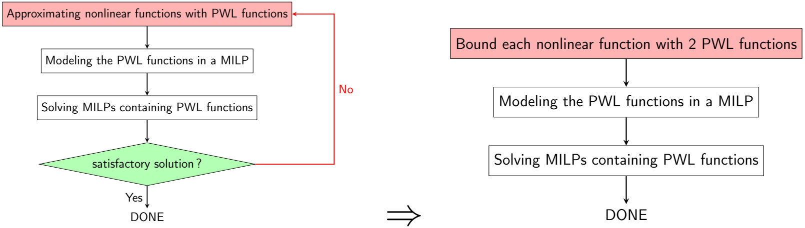
Fitting a discrete data set

Applications : ECG, pattern recognition, data reduction, ...



Bellman and Roth 1969 ... Tomek 1974 ... O'Rourke 1981 ... Toriello and Vielma 2012 ... Rebennack and Krasko 2019

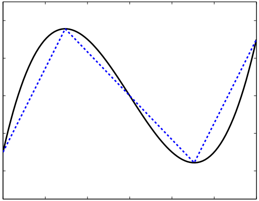
From an iterative to a non-iterative solution method ?



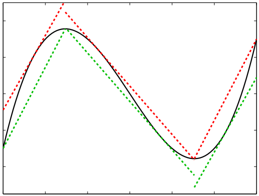
New two-step solution scheme : Ngueveu et al., 2016, 2019

PGMO projects OREM (2014-2016), OPAL(2016-2018)

Step 1 : Piecewise linear bounding of the nonlinear energy transfer/efficiency functions



(c) Linear approximation



(d) Piecewise bounding

Step 2 : Reformulation of the problem into two mixed integer problems (MILP)

- solve with a MILP solver
- or design a dedicated solution method (only one needed)

Resulting PWL approximation/bounding Problem

Given a function $f : \mathbb{R} \to \mathbb{R}$ and a tolerance bound, find a PWL function that :

- Respects the imposed bounded error
- Minimizes the number of pieces of the PWL function

Benefits

- predetermined error criteria $\rightarrow$ non-iterative approach

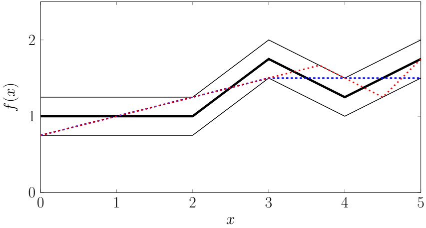
  Bound each nonlinear function with 2 PWL functions
  ↓
  Modeling the PWL functions in a MILP
  ↓
  Solving MILPs containing PWL functions
  ↓
  DONE

- Smallest MILP possible !

Why is the Problem Hard ?

Bounded tolerance constraints → semi-infinite programming (SIP)

Maximizing the length of the projection on x-axis of each segment is not optimal



Approximation of continuous fonctions with PWL functions

Minimize number of linear pieces for a bounded error or distance metric

Rosen and Pardalos, 1986. Global minimization of large-scale constrained concave quadratic problems by separable programming. **Math. Prog**

Frenzen, Sasao and Butler, 2010. On the number of segments needed in a piecewise linear approximation. **J. of Comp. and Applied Math.**

Geibler, Martin, Morsi, Schewe, 2012. Using piecewise linear functions for solving MINLPs. **The IMA Volumes in Mathematics and its applications**

Rebennack and Kallrath, 2015. Continuous piecewise linear δ-approximations for univariate functions : computing minimal breakpoint system. **J. Optim. Theory Appl**

Ngueveu, 2019. Piecewise linear bounding of univariate nonlinear functions and resulting mixed integer linear programming-based solution methods **EJOR**

Rebennack and Krasko, 2019. Piecewise linear function fitting via mixed-integer linear programming. **INFORMS Journal on Computing**

Continuous PWL approximation of univariate functions

Optimal Breakpoint System using a Continuum approach for x [RK2015]

Decision variables

- $x_b \in [X_-, X_+]$: breakpoint value
- $s_b \in [-\delta, +\delta]$: deviation on bpt b
- $\chi_b \in [0,1] = 1$ iff bpt b is used
- $y_b \geq \frac{1}{M} := x_b - x_{b-1}$ if $x_b - x_{b-1} > 0$ and $= |X_-, X_+|$ otherwise
- $\xi^x_{bx} \in [0,1]$: =1 iff $x \in [x_{b-1}, x_b]$
- $l_b(x) \in \mathbb{R} := l(x)$ if $x \in [x_{b-1}, x_b]$

Semi-infinite programming model

Obj = minimize number of active breakpoints
s.t. (c1) Order active breakpoints
(c2) Link x_b and χ_b
(c3) Compute y_b and $l_b(x)$
(c4) Compute ξ^x_{bx}
(c5) Compute $l(x)$
(c6) Ensure the δ-approximation : $|l(x) - f(x)| \leq \delta, \forall x \in D$

Continuous PWL approximation of univariate functions

Solution 1 : Semi-infinite programming models (large and difficult)

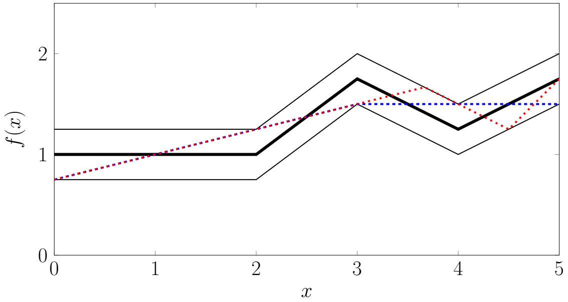
Iterative solution method

- enforce the δ gap constraint on discrete points $|\mathbb{I}|$
  - Rebennack and Kallrath 2015 $\Rightarrow$ solve a MINLP
  - Rebennack and Krasko 2019 $\Rightarrow$ solve a MILP (extension of Toriello and Vielma 2012)
- compute the real maximum error
  - Rebennack and Kallrath 2015, 2019 $\Rightarrow$ solve an NLP

Continuous PWL approximation of univariate functions

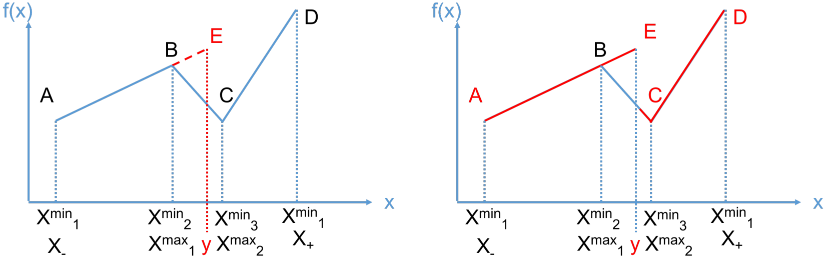
Solution 2 : Greedy heuristics computing x_b given x_{b-1},s_{b-1} and δ

- Max length of interval $[x_{b-1}, x_b]$(projection on x-axis) : not optimal



- solve the NLP problem of max x_b or try discrete values of x_b and s_b

Non necessarily continuous (nnc) PWL approximation



Proposition

Any optimal continuous δ-PWLA with n^* pieces can be converted into a non-necessarily continuous δ-PWLA with $n \leq n^*$ pieces where the projection of the first piece on interval D is of maximal length

Non necessarily continuous (nnc) δ-PWLA

Getting rid of the continuity

Theorem

$\forall$ continuous function $f : \mathbb{D} = [X_-, X_+] \to \mathbb{R}$ and any scalar $\delta \in \mathbb{R}^+$, there exists an optimal nnc δ-PWLA g defined by $G = \bigcup_{i=1}^{n_g}([a_i, b_i], [x_i^{\min}, x_i^{\max}])$ such that each line-segment i has a maximal length projection on the interval $[x_i^{\min}, X_+]$.

The greedy algorithm becomes optimal

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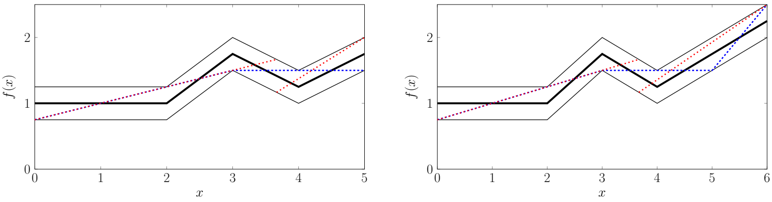


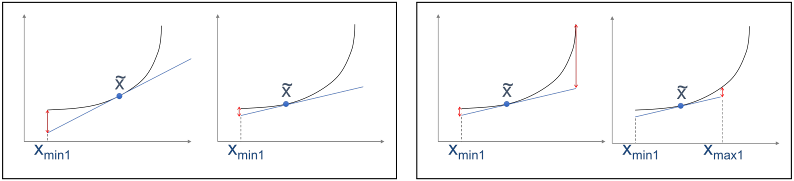
$(n^* + 1)/2 \leq n_g \leq n^*$

Better δ-PWL approximation algorithms

From δ-approx to $\frac{\delta}{2}$-under(-over)estimator : shift by $\frac{\delta}{2}$

Convex or concave function

optimal 2-steps dichotomic search : No NLP or MINLP solved !



step a : max $\tilde{x}$ step b : max $x_{\max_i}$

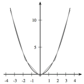
Piecewise convex or concave function with p pieces

approx/bound each convex or concave piece separately

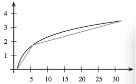
$$n_g^* \leq n_g \leq n_g^* + p$$

Numerical comparisons Computational evaluation on continuous functions Computational evaluation on continuous functions

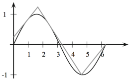
Computational evaluation on continuous functions



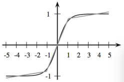
x^2



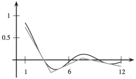
$\ln(x)$



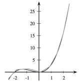
$\sin(x)$



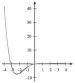
$\tanh(x)$



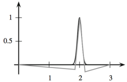
$\frac{\sin(x)}{x}$



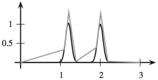
$2x^2 + x^3$



$e^{-x}\sin(x)$



$e^{-100(x-2)^2}$



$1.03e^{-100(x-1.2)^2} + e^{-100(x-2)^2}$

f		δ	continuous approx. [RK2015]		nnc approximation Exact		Heuristic			
	p		n_*	cpu	n_*	cpu	n_*	n_-	n_+	cpu
I	1	0.01	**25**	Hours	**25**	19 s	**25**			30 s
		0.005	**35**	Hours	**35**	4 s	**35**			43 s
II	1	0.01	**9**	Sec	**9**	221 s	**9**			11 s
		0.005	**13**	Sec	**13**	172 s	**13**			16 s
III	2	0.01	**13**	Sec	**13**	57 s		13	14	
		0.005	**17**	Few min	**17**	68 s		17	18	
IV	2	0.01	**9**	Few sec	**9**	161 s		9	10	10 s
		0.005	**13**	Few min	**13**	128 s		13	14	15 s
V	4	0.01	9	Sec	**8**	143 s		7	10	11 s
		0.005	**12**	Few min	**12**	181 s		12	15	15 s

f		δ	continuous approximation [RK2015]				nnc approximation [Ng2019] Exact		Heuristic		
	p		n_*	n_-	n_+	cpu	n_*	cpu	n_-	n_+	cpu
VI	2	0.05	**15**			Few days	**15**	88 s	15	16	17 s
		0.01		15	**34**		**34**	164 s	34	35	36 s
		0.005		15	**47**		**47**	195 s	47	48	59 s
VII	3	0.05		4	**19**		**19**	287 s	19	21	28 s
		0.01		4	**43**		**43**	268 s	43	45	52 s
		0.005		4	**61**		**61**	869 s	61	63	72 s
VIII	3	0.05		4	6		**5**	83 s	4	6	6 s
		0.01		4	**11**		**11**	138 s	10	12	11 s
		0.005		4	**14**		**14**	1466 s	14	16	16 s
IX	5	0.05		7	11		**9**	64 s	7	11	12 s
		0.01		7	**21**		**21**	114 s	19	23	23 s
		0.005		7	28		**27**	784 s	27	31	27 s

f		δ	continuous approximation [RK2019]			nnc approximation [Ng2019] Exact		Heuristic		
	p		n_*	n_-	cpu	n_*	cpu	n_-	n_+	cpu
VI	2	0.05	**15**		107.7 s	**15**	88 s	15	16	17 s
		0.01		31	6819.1 s	**34**	164 s	34	35	36 s
		0.005		40	35787.4 s	**47**	195 s	47	48	59 s
VII	3	0.05	**19**		14514.6 s	**19**	287 s	19	21	28 s
		0.01		34	35411.0 s	**43**	268 s	43	45	52 s
		0.005		35	70313.1 s	**61**	869 s	61	63	72 s
VIII	3	0.05	**5**		5.3 s	**5**	83 s	4	6	6 s
		0.01	**11**		59.6 s	**11**	138 s	10	12	11 s
		0.005	**14**		247.2 s	**14**	1466 s	14	16	16 s
IX	5	0.05	**9**		12 s	**9**	64 s	7	11	12 s
		0.01	**21**		13873.4 s	**21**	114 s	19	23	23 s
		0.005	**27**		42068.4 s	**27**	784 s	27	31	27 s

Weaknesses/Limitations of PWL-based solution methods

- dependent on the instance : e.g. dependent on time horizon
- dependent on the solution : e.g. solution cost
- difficult to choose a relevant value of δ for a new instance or worse a new problem

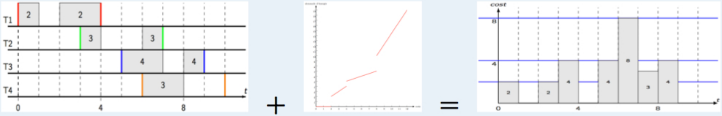
Energy optimization in hybrid electric vehicle

$$\min \sum_{i=1}^{n} f_{\mathrm{FC}}(x_i) \tag{11}$$

s.t.
(c1) Power demand satisfaction
(c2) Final SE energy level higher or equal to initial energy level
(c3) SE energy level within bounds
(c5) Variables domains

Weaknesses/Limitations of PWL-based solution methods

Step 2 : Pre-emptive scheduling problem with non linear efficiency functions



(c) Scheduling + (d) Efficiency = (e) Energy cost

Minimize the total energy cost

$$(\mathrm{CF})\ \min \quad F = \sum_{t \in T} f\Big(\sum_{i \in \mathcal{A}} b_i x_{it}\Big) \tag{12}$$

s.t. Satisfaction of the demand for each activity

$$\sum_{t \in T} a_{it} x_{it} \geq p_i, \qquad \forall i \in \mathcal{A} \tag{13}$$

Validity domain

$$x_{it} \in \{0, 1\}, \qquad \forall i \in \mathcal{A}, t \in T \tag{14}$$

Use relative ϵ-tolerance

Principle

For a function $f : \mathbb{D} \to \mathbb{R}^*$ and a tolerance value $\epsilon \in [0, 1]$, identify two piecewise linear functions $(\overline{f}^{\epsilon}, \underline{f}^{\epsilon})$ that verify :

$$\underline{f}^{\epsilon}(x) \leq f(x) \leq \overline{f}^{\epsilon}(x), \quad \forall x \in \mathbb{D} \tag{15}$$

$$|f(x) - \underline{f}^{\epsilon}(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \tag{16}$$

$$|\overline{f}^{\epsilon}(x) - f(x)| \leq \epsilon |f(x)|, \quad \forall x \in \mathbb{D} \tag{17}$$

Purpose

- Apply on nonlinear term BEFORE insertion in the model
  - univariate function that can be convex or concave, easier to bound
  - minimize number of pieces ($\neq$ spatial or interval branch-and-bound)
- Two MILP ($\overline{\mathrm{MILP}}$ and $\underline{\mathrm{MILP}}$) are obtained
- Guarantees on the quality of the resulting lower and upper bounds
- Upper and lower bounding PWL functions $\overline{f}^{\epsilon}(x)$ and $\underline{f}^{\epsilon}(x)$ may not share the same breakpoints and number of pieces (no shift !)

Numerical comparisons

Comparison on the energy optimization problem for hybrid electric vehicles

- upper bounds
- lower bounds
- absolute vs relative tolerance

Comparison of upper bounds

Instances			ϵ-PWLB+MILP [Ng2019]			Effpts+MILP [GCL2013]	
class	#	$\overline{n}^{\epsilon}$	gap orig	cpu	gap recomp	gap	cpu
(R)	6	6	0.20 %	16 s	0.04 %	1.07 %	3000 s
		56	**0.01** %	58 s	**0.01** %	0.03 %	314 s
(A1)	6	10	0.26 %	31 s	0.18 %	0.62 %	3000 s
		82	**0.01** %	78 s	**0.01** %	0.03 %	488 s
(A2)	6	14	0.52 %	96 s	0.49 %	2.26 %	2436 s
		133	**0.01** %	627 s	**0.01** %	0.07 %	1213 s

Instances			antigone	baron	couenne	lindoglobal
class	#	optCR	gap	gap	gap	gap
(R)	6	1 %	13.86 %	9.64 %	20.97 % (4)	0.01 % (5)
		0.01 %	3.78 %	9.69 %	5.40 % (4)	0.01 % (5)
(A1)	6	1 %	1.06 %	8.62 %	3.44 % (5)	0.37 % (5)
		0.01 %	1.71 %	10.15 %	3.44 % (5)	0.37 % (5)
(A2)	6	1 %	0.96 %	14.74 %	20.11 % (2)	0.00 % (5)
		0.01 %	0.94 %	14.73 %	20.11 % (2)	0.00 % (5)

Comparison of lower bounds

Instances		old lower bound [NCMG2019]		ϵ-PWLB+MILP [Ng2019]		
set	#	ratio	cpu	ϵ	ratio	cpu
(R)	6	97.81 %	2 s	1 %	99.18 %	20 s
				0.01 %	**99.99** %	58 s
(A1)	6	97.36 %	2 s	1 %	99.25 %	29 s
				0.01 %	**99.99** %	216 s
(A2)	6	78.50 %	2 s	1 %	99.50 %	26 s
				0.01 %	**99.99** %	405 s

Instances			antigone	baron	couenne	lindoglobal
class	#	optCR	ratio	ratio	ratio	ratio
(R)	6	1 %	24.66 %	- % (6)	- % (6)	- % (6)
		0.01 %	21.32 %	- % (6)	- % (6)	- % (6)
(A1)	6	1 %	49.41 %	37.89 % (0)	33.03 % (3)	35.99 % (5)
		0.01 %	49.60 %	37.72 % (0)	33.07 % (3)	35.80 % (5)
(A2)	6	1 %	53.21 %	- % (6)	- % (6)	- % (6)
		0.01 %	53.21 %	- % (6)	- % (6)	- % (6)

Comparison of relative vs absolute tolerance

Class (R)	using relative tol			using absolute tol		Gap
instance	ϵ	n^ϵ	UB	δ	n^δ	$\frac{n^\delta - n^\epsilon}{n^\epsilon}$
S_40	1 %	6	454.3	0.1136	14	133.33 %
	0.01 %	56	453.4	0.0011	133	137.50 %
I_561	1 %	6	8756.6	0.1561	12	100.00 %
	0.01 %	56	8741.0	0.0016	110	96.43 %
H_734	1 %	6	18626.0	0.2538	10	66.67 %
	0.01 %	56	18569.0	0.0025	89	58.93 %
U_811	1 %	6	2613.5	0.0322	25	316.67 %
	0.01 %	56	2607.7	0.0003	255	355.36 %
N_1200	1 %	6	23137.7	0.1928	11	83.33 %
	0.01 %	56	23114.9	0.0019	102	82.14 %
E_1400	1 %	6	27088.9	0.1935	11	83.33 %
	0.01 %	56	27065.9	0.0019	102	82.14 %

58% to 355% more linear pieces with absolute vs relative tolerance

PWL bounding with relative ϵ-tolerance

Strength

- efficiency for problems with a nonlinear function per data set
- various fields and domains of application

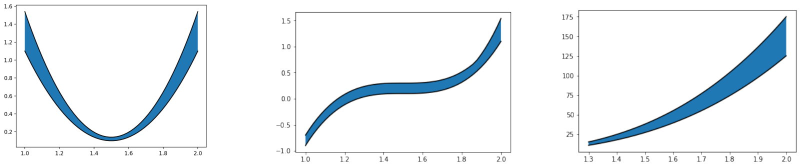
To be improved

- access/ease-of-use for non-technical users
- speed improvement to tackle problems with multiple nonlinear functions

New perspective

Definition (Corridor)

Let $h, l : [a,b] \to \mathbb{R}$ C^1 $h(x) \geq l(x), \forall x \in [a,b]$ and having the same concavity. We call $\mathcal{C} = \{(x,y) | x \in [a,b], l(x) \leq y \leq h(x)\}$ a corridor between h and l .



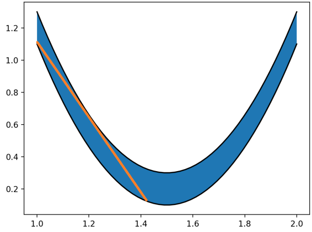
Codsi, Gendron, Ngueveu (2019)

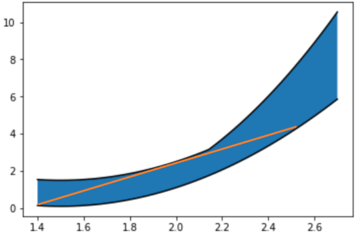
Convex corridor

Theorem (Convex corridor segment characterization)

On convex corridor $\mathcal{C}$ there exists an optimal linear segment such that

- *Both ends lie on the lower curve*
- *it is tangent to the upper curve*



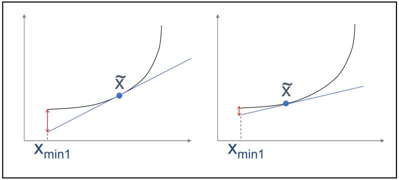


Convex corridor algorithm

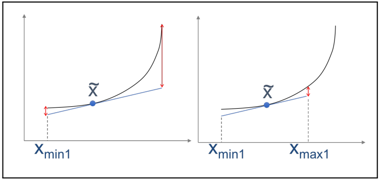
generalisation of the algorithm of Ngueveu 2019

- Use dichotomy to find the tangent point
- find the resulting segment
- repeat 1 and 2 for the next segment

optimal 2-steps dichotomic search : No NLP or MINLP solved !



step a : max $\tilde{x}$



step b : max x_{max_i}

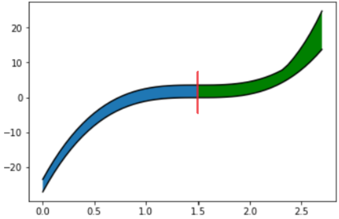
Strength of the algorithm

- logarithmic convergence (for each segment)
- Works on concave corridors too
- not limited to absolute error
- can be used to approximate, underestimate and overestimate functions

Corridors Without Constant Convexity

Splitting the corridor

+ parallelizable
+ Efficient
- Heuristic Not necessarily optimal but the error is tightly bounded



$$n^* \leq n \leq n^* + \#\text{Sub-corridors}$$

O'Rourke adaptation

based on function sampling and constraints on the line coefficient space

+ Exact
- Not as efficient

function	absolute error	Continuous		ncc			
		time					
		exact		exact		Heuristic	
		[RK15]	[RK19]	[Ngu19]	New	[Ngu19]	New
VI	0.1	Min	24.4 s	115 s	2.9 s	11 s	0.8 s
	0.05	Few days	107.7 s	88 s	3.0 s	17 s	0.8 s
	0.01	*	*	164 s	3.5 s	36 s	0.8 s
	0.005	*	*	195 s	2.8 s	59 s	0.8 s
VII	0.1	*	311.1 s	226 s	4.0 s	17 s	0.7 s
	0.05	*	14 514.6 s	287 s	3.9 s	28 s	0.8 s
	0.01	*	*	268 s	4.9 s	52 s	0.9 s
	0.005	*	*	869 s	5.0 s	72 s	0.9 s
VIII	0.1	sec	1.7 s	74 s	4.6 s	4 s	0.4 s
	0.05	*	5.3 s	83 s	5.4 s	6 s	0.4 s
	0.01	*	59.6 s	138 s	4.7 s	11 s	0.4 s
	0.005	*	247.2 s	1466 s	4.5 s	16 s	0.4 s
IX	0.1	Few days	1.9 s	77 s	8.3 s	8 s	0.8 s
	0.05	*	12 s	64 s	8.1 s	12 s	0.8 s
	0.01	*	13 873.4 s	114 s	8.1 s	23 s	0.8 s
	0.005	*	42 068.4 s	784 s	8.1 s	27 s	0.7 s

Results

Network design with congestion

- from a multivariate objective-function ... $\min \sum_{(i,j)\in A} O_{ij} y_{ij} + \sum_{(i,j)\in A} \sum_{p\in P} D_{ij}^p x_{ij}^p + \sum_{i\in N} E_i z_i + \sum_{i\in N} g_i(\sum_{j\in N} \sum_{p\in P} x_{ij}^p, z_i)$
- ... to a univariate objective-function $\min \sum_{(i,j)\in A} O_{ij} y_{ij} + \sum_{(i,j)\in A} \sum_{p\in P} D_{ij}^p x_{ij}^p + \sum_{i\in N} f_i(v_i)$

instance	litterature	new
c35_0.3_0.6	6,615 s	6,92 s
c36_0.8_0.8	21,528 s	14,59 s
c49_0.8_0.6	172,255 s	118,17 s
c50_0.8_0.6	2609,568 s	2575,08 s

- as good as advanced state of the art solution methods
- no consideration on the problem structure
- easy to implement

Implementation

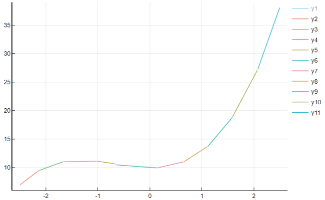
An approximation package named JULIA

Implementing both the exact and heuristic methods and include many classical error metrics !

- Julia based

$2x^2 + x^3 + 10$

```
linearApprox(:(2x^2+x^3+10),-2.5, 2.5, relative = true, epsilon = 1.0)
```



With an error guarantee !

Conclusion

Done

- PWLA+MILP efficient for solving certain classes of MINLPs
  - Non-necessarily continuous piecewise linear functions
  - Relative $\epsilon-$ tolerance
  - Bounding instead of approximation
- 2 similar MILPs to solve
- Various applications

What next ?

- Coming soon (next few days) : opensource PWLB toolbox
- Extension to non-separable functions
- Other classes of problems ? (stochastic programming ?)

Bound each nonlinear function with 2 PWL functions

↓

Modeling the PWL functions in a MILP

↓

Solving MILPs containing PWL functions

↓

DONE

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Useful Tools / Julia Packages

LINA : Computing a PWL approximation, over-/under-estimators with minimum # linear segments

- link : `http://homepages.laas.fr/sungueve/LINA.html`
- input : a univariate continuous nonlinear function
- output : a nnc PWL function with minimum number of pieces
- related reference : Codsi, Gendreau, Ngueveu (2019-HAL)

PiecewiseLinearOpt : Modeling efficiently a given continuous PWL function in MILP

- `https://github.com/joehuchette/PiecewiseLinearOpt.jl`
- input : a continuous PWL function (or sampled nonlinear fct)
- output : variables and constraints to insert in a MILP
- related reference : Huchette and Vielma (2018-arXiv)