Stochastic Vehicle Routing: an Overview and an Exact Solution Approach for the Vehicle Routing Problem with Stochastic Demands under an Optimal Restocking Recourse Policy

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Outline

1. Introduction
2. Basic concepts in stochastic optimization
3. Modeling paradigms
4. VRP with stochastic demands
5. Some other recourse policies for the VRPSD
6. An exact algorithm to solve the VRPSD under an optimal restocking policy
7. Conclusion and perspectives
Introduction
Vehicle Routing Problems

- Introduced by Dantzig and Ramser in 1959
- One of the most studied problem in the area of logistics
- The basic problem involves delivering given quantities of some product to a given set of customers using a fleet of vehicles with limited capacities.
- The objective is to determine a set of minimum-cost routes to satisfy customer demands.
Many variants involving different constraints or parameters:

- Introduction of travel and service times with route duration or time window constraints
- Multiple depots
- Multiple types of vehicles
- ...

Vehicle Routing Problems
What is Stochastic Vehicle Routing?

Basically, any vehicle routing problem in which one or several of the parameters are not deterministic:

- Demands
- Travel or service times
- Presence of customers
- ...
Main classes of stochastic VRPs

- **VRP with stochastic demands (VRPSD)**
  - A probability distribution is specified for the demand of each customer.
  - One usually assumes that demands are independent (this may not always be very realistic...).

- **VRP with stochastic customers (VRPSC)**
  - Each customer has a given probability of requiring a visit.

- **VRP with stochastic travel times (VRPSTT)**
  - The travel times required to move between vertices, as well as sometimes service times, are random variables.
Basic Concepts in Stochastic Optimization
Dealing with uncertainty in optimization

- Very early in the development of operations research, some top contributors realized that:
  - In many problems there is very significant uncertainty in key parameters;
  - This uncertainty must be dealt with explicitly.
- This led to the development of:
  - Stochastic programming with recourse (1955)
  - Dynamic programming (1958)
  - Chance-constrained programming (1959)
  - Robust optimization (more recently)
Information and decision-making

In any stochastic optimization problem, a key issue is:

- How do the revelation of information on the uncertain parameters and decision-making (optimization) interact?
  - When do the values taken by the uncertain parameters become known?
  - What changes can I (must I) make in my plans on the basis of new information that I obtain?
Stochastic programming with recourse

- Proposed separately by Dantzig and by Beale in 1955.
- The key idea is to divide problems in different stages, between which information is revealed.
- The simplest case is with only two stages. The second stage deals with **recourse actions**, which are undertaken to adapt plans to the realization of uncertainty.

Basic reference:

Dynamic programming

- Proposed by Bellman in 1958.
- A method developed to tackle effectively sequential decision problems.
- The solution method relies on a time decomposition of the problem according to stages. It exploits the so-called Principle of Optimality.
- Good for problems with limited number of possible states and actions.
- Basic reference:
Chance-constrained programming

- Proposed by Charnes and Cooper in 1959.

- The key idea is to allow some constraints to be satisfied only with some probability.

E.g., in VRP with stochastic demands,

\[ \Pr\{\text{total demand assigned to route } r \leq \text{capacity} \} \geq 1-\alpha \]
Robust optimization

- Here, uncertainty is represented by the fact that the uncertain parameter vector must belong to a given polyhedral set (without any probability defined)
  - E.g., in VRP with stochastic demands, having set upper and lower bounds for each demand, together with an upper bound on total demand.

- Robust optimization looks in a minimax fashion for the solution that provides the best “worst case”.
Modelling Paradigms
Real-time optimization

Also called re-optimization

- Based on the implicit assumption that information is revealed over time as the vehicles perform their assigned routes.
- Relies on Dynamic programming and related approaches (Secomandi et al.)
- Routes are created piece by piece on the basis on the information currently available.
- Not always practical (e.g., recurrent situations)
A priori optimization

- A solution must be determined beforehand; this solution is “confronted” to the realization of the stochastic parameters in a second step.

- Approaches:
  - Chance-constrained programming
  - (Two-stage) stochastic programming with recourse
  - Robust optimization
  - [“Ad hoc” approaches]
Chance-constrained programming

- Probabilistic constraints can sometimes be transformed into deterministic ones (e.g., in VRP with stochastic demands, when one imposes that
  \[ \Pr\{\text{total demand assigned to route } r \leq \text{cap.} \} \geq 1-\alpha, \]
  if customer demands are independent and Poisson).

- This model completely ignores what happens when things do not “turn out correctly”.
Robust optimization

- Not used very much in stochastic VRP up to now.

- Model may be overly pessimistic.
Recourse is a key concept in a priori optimization
- What must be done to “adjust” the a priori solution to the values observed for the stochastic parameters?
- Another key issue is deciding when information on the uncertain parameters is provided to decision-makers.

Solution methods:
- Integer L-shaped (Laporte and Louveaux)
- Column generation (Branch & Price)
- Heuristics (including metaheuristics)

Probably closer to actual industrial practices, if recourse actions are correctly defined!
VRP with Stochastic Demands
VRP with stochastic demands

- Probably, the most studied stochastic vehicle routing problem.
- A variant of the well-known CVRP in which the customer demands are not fixed quantities, but instead random variables with given distributions.
- Building solutions in which capacity constraints would be satisfied for all possible realizations of the customer demands is clearly too conservative.
We must consider solutions in which the total demand of customers assigned to a given route can possibly exceed the capacity of a vehicle.

What should be done if this happens ?!??

We must engage in some corrective or RECOURSE action.
Classical recourse

- It is important to understand that recourse actions should reflect operating policies of the company operating the fleet of vehicles.

- The classical recourse strategy used in most of the VRPSD literature consists of returning to the depot to restore the vehicle capacity and then resuming the planned route from the point of failure.

- This strategy allows for an “easy” computation of the expected recourse associated with any given route.
Other possible recourses

- The classical recourse strategy is not very bright. It is even shocking to some people who are not used to it.
- Yang, Mathur, and Ballou (2000) have proposed a policy of *optimal restocking* of the vehicle based on dynamic programming, but it can only be applied to given routes.
- Ak and Erera (2007) have suggested “pairing” to enhance operations and reduce the number of back and forth trips back to the depot.
VRP with stochastic demands

- Approximate solutions can be obtained fairly easily using metaheuristics (e.g., Tabu Search, as in Gendreau et al., 1996).

- Computing effectively the value of the recourse function still remains a challenge.
A branch-and-cut approach (direct formulation)

The following material is taken from

Model

Input

\[ G(V, E) = \text{undirected graph, } V = \{v_1, \ldots, v_n\} \text{ and } E = \{(v_i, v_j): v_i, v_j \in V, i < j\} \]
\[ \xi_i = \text{demand of client } i, \text{ where } i = 2, \ldots, n \]
\[ C = (c_{ij}) \text{ travel cost matrix} \]
\[ D = \text{capacity of each vehicle} \]

Decision variables

\[ x_{ij} = \{0, 1\} \text{ for } i, j > 1 \]
\[ x_{1j} = \{0, 1, 2\} \text{ for } j > 1 \]

The considered case

Client demands are independent
\[ \xi_j \sim N(\mu_j, \sigma_j) \text{ and } \xi_j \in (0, D), j = 2, \ldots, n \]
Recourse rules ⇒ return to depot only when failure occurs
For a given route \((v_{r1} = v_1, v_{r2}, \ldots, v_{rt+1} = v_1)\)

\[ Q^{1,r} = 2 \sum_{i=2}^{t} \sum_{l=1}^{i-1} P \left( \sum_{s=2}^{i-1} \xi_{rs} \leq lD < \sum_{s=2}^{i} \xi_{rs} \right) c_{1r_i} \]
\[ Q(x) = \sum_{i=1}^{m} \min \{Q_{k,1}^{k}, Q_{k,2}^{k}\}, \]
**Model**

VRPSD (Virtual Reality Planning and Scheduling Domain) model

Minimize $\sum_{i<j} c_{ij} x_{ij} + Q(x)$

subject to

$$\sum_{j=2}^{n} x_{1j} = 2m,$$

$$\sum_{i<k} x_{ik} + \sum_{j>k} x_{kj} = 2, \quad (k = 2, \ldots, n),$$

$$\sum_{v_i, v_j \in S} x_{ij} \leq |S| - \left[ \sum_{v_i \in S} \mathbf{E}(\xi_i)/D \right], \quad S \subset V \setminus \{v_1\}, 2 \leq |S| \leq n - 2$$

$$0 \leq x_{ij} \leq 1 \quad 1 \leq i < j < n),$$

$$0 \leq x_{0j} \leq 2 \quad (j = 2, \ldots, n),$$

$$x = (x_{ij}) \quad \text{integer.}$$
A branch-and-cut approach

- Computational results on problems with independent truncated Normal demands
  - 30 instances for each combination of $m$ and $n$
  - 10 hours of CPU time for each run.

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A branch-and-price approach
(set covering formulation)

The following material is taken from

Introduction

Literature

- **Heuristics**
  - Tillman (1965) - Savings-based heuristics for multi-depot and Poisson demands
  - Golden, Stewart (1983)
  - Gendreau, Séguin (1996) - Tabou search

- **Integer L-shaped method for problems with simple recourse**
  - Laporte, Louveaux (1993)
  - Hjorring, Holt (1999)
  - Laporte, Louveaux, Van Hamme (2001)

- **Column Generation**
  - Christiansen, Lysgaard (2007) *Branch-and-price*

Our Contribution

Develop a competitive *branch-cut-and-price algorithm* for the VRPSD based on the work of Christiansen et Lysgaard (2007).
G = (N′ = N ∪ {o, o′}, A) : undirected graph
N′ = {1, 2, ..., n} : set of clients
A = {(i, j)|i, j ∈ N; i ≠ j} ∪ {(o, j)|j ∈ V} ∪ {(j, o′)|j ∈ N} : set of arcs
**Notation**

- $c_{ij}$: determinist travel cost from $i \in \mathcal{N}'$ to $j \in \mathcal{N}'$
- $p = (i_1, \ldots, i_h)$: route where $i_j \in \mathcal{N}'$ for $j \in \{1, \ldots, h\}$
- $c_p$: determinist travel cost of route $p$
- $\hat{c}_p$: total expected failure cost of route $p$
- $\alpha_{ip}$: binary parameter indicating if route $p$ visits client $i \in \mathcal{N}$ or not
- $\mathcal{P}$: set of all routes from $o$ to $o'$ which are feasible on average
Notation

- $\mathcal{V}$ : set of $|\mathcal{V}|$ identical vehicles
- $Q$ : capacity of a vehicle
Notation

- $\xi_i \sim \Psi(E[\xi_i], V[\xi_i])$: random variable indicating demand of client $i \in \mathcal{N}$ following distribution $\Psi$;

- Given path $p = (i_1, i_2, \ldots i_h)$
  - $\sum_{j=1}^{h} E[\xi_{ij}] = \mu_{ih}$: expected cumulated demand at $i_h$
  - $\sum_{j=1}^{h} V[\xi_{ij}] = \sigma_{ih}$: expected cumulated variance at $i_h$
Notation - decision variables

First stage
- $x_{ij}$: binary variable indicating if a vehicle follows the arc $(i, j) \in A$.
- $\lambda_p$: binary variable indicating if we choose route $p$ or not.

Second stage
- $Q(x)$: recourse function (expected failure cost of a given route).
Master Problem and Subproblem - VRPSD

\[
\begin{align*}
\min_{\lambda} & \quad \sum_{p \in \mathcal{P}} \hat{c}_p \lambda_p \\
\text{s. t.} & \quad \sum_{p \in \mathcal{P}} \alpha_{ip} \lambda_p = 1 \forall i \in \mathcal{N} \\
& \quad \lambda_p \in \{0, 1\} \quad \forall p \in \mathcal{P} \\
& \quad \lambda_p \in \{0, 1\} \quad \forall p \in \mathcal{P}
\end{align*}
\]

\[(MP)\]

\[
\begin{align*}
\min_{x} & \quad \sum_{i \in \mathcal{N}'} \sum_{j \in \mathcal{N}'} \bar{c}_{ij} x_{ij} + Q(x) \\
\text{s. à :} & \quad \sum_{(i,j) \in \delta^- \setminus \{j\}} x_{ij} - \sum_{(j,i) \in \delta^+ \setminus \{j\}} x_{ji} = \begin{cases} 
-1 & \text{if } j = o \\
0 & \text{else } \forall j \in \mathcal{N}' \\
1 & \text{if } j = o'
\end{cases} \\
& \quad \sum_{i \in \mathcal{N}'} \sum_{j \in \mathcal{N}'} \mathbb{E}_{\xi} [\xi_{ij}] x_{ij} \leq Q \\
& \quad x_{ij} \in \{0, 1\} \quad \forall i, j \in \mathcal{N}'
\end{align*}
\]

\[(SP)\]

Reduced cost

\[
\bar{c}_{ij} = c_{ij} - \pi_j \quad \text{if } j \neq o'
\]
Given

\[ p = (i_1, i_2, \ldots, i_{h-1}, i_h) \subseteq \mathcal{N}' \text{ from } o = i_1 \text{ to } i_h \]

**1. Expected failure cost at client \( i_h \) for \( p \)**

\[
\text{EFC}(\mu_{i_h}, \sigma_{i_h}, i_h) = 2c_{oi_h} \sum_{u=1}^{\infty} (\mathbb{P}_\xi \left[ \sum_{l=1}^{h-1} \xi_i \leq uQ < \sum_{l=1}^{h} \xi_i \right])
\]


**Total cost of route \( p \):**

\[
\hat{c}_p = \sum_{j=1}^{h} \left( c_{i_hi_{h+1}} + \text{EFC}(\mu_{i_{h+1}}, \sigma_{i_{h+1}}, i_{h+1}) \right)
\]
Creation of the state-space graph \( G = (\mathcal{N}, \mathcal{A}) \)

\[
0 \xrightarrow{c_{oj} + EFC(\mu_j, \sigma_j)} 0' \\
(\mu_i + E[\xi_j], \sigma_i + V[\xi_j]) \xrightarrow{(E[\xi_j], V[\xi_j])} \ldots \\
(\mu_i + E[\xi_j], \sigma_i + V[\xi_j]) \xrightarrow{(E[\xi_i + \xi_j], V[\xi_i + \xi_j])} \ldots \\
(\mu_i + E[\xi_j], \sigma_i + V[\xi_j]) \xrightarrow{(Q, V[\xi_j])} \ldots \\
(\mu_i + E[\xi_j], \sigma_i + V[\xi_j]) \xrightarrow{(Q, V_{max}[\xi_j])} \ldots \\
\mu_j = Q
\]
Max allotted time : 20 minutes, 
3,4 GHz and 16 GB of RAM vs 1,5GHz and 480 MB of RAM ⇒ scaling factor : 1.98

Poisson demands + A, E, P instances

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<th>Instance</th>
<th>Base (This work)</th>
<th>Christiansen &amp; Lysgaard (2007)</th>
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### Numerical results (cont’d)

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A robust optimization approach


- The authors consider the generic case where the customer demands are supported on a polyhedron.
- Robust solutions must be feasible for all demand vectors in the polyhedron.
- Several interesting references.
Some other Recourse Policies for the VRPSD
Other possible recourses

- One can easily imagine other recourse policies, which would be easily implementable in practice, and which would make more sense.

- This is what we did in the two first papers of Majid Salavati-Khosgalb’s dissertation:

Stochastic VRP : An Overview and an Exact Method for Optimal Restocking
Fixed-policy recourse

- Each fixed policy can be derived from a given operational convention.

- It generates related policy-based recourse actions, which are defined as a set of thresholds associated with the customer visits that are scheduled along a given route.

- These thresholds determine when the vehicle performing the route should preventively return to the depot.
Four classes of pro-active policies

- *Demand-based policies*, in which thresholds relate directly to the demand levels of customers or the capacity of the vehicle.
- *Risk-based policies*, in which thresholds are computed on the basis of the probability of failure at the next or at following customers.
- *Distance-based policies* also account for the distance between customers and the depot.
- *Hybrid (or mixed) policies* combine features of two or three of the other classes.
Three demand-based policies

- The $\alpha Q$ policy is defined in terms of the vehicle capacity $Q$: PR trips occur whenever the residual capacity falls below a fraction $\alpha$ of $Q$.
  - Three values are examined for $\alpha$: 0.05, 0.10, and 0.20. In this case, all thresholds are equal.

- In the second policy considered, the threshold $\theta_{ij}$ is a function of the expected demand of the following customer $v_{ij+1}$.
  - Three values are considered for $\beta = 0.80, 1.00, 1.25$.

- The third policy is based on the expected demand of the customers remaining on the route after $v_{ij}$.
A mixed policy

- We consider a mixed policy that combines the basic structure of a risk-based policy with a distance-based policy.

- After serving each customer, the risk of failure at the next customer is computed,
  - If the risk is high: perform a preventive restocking trip;
  - If the risk is low: proceed;
  - If the risk is medium: apply a distance-based policy.

- The distance-policy criterion is to compare the cost of a preventive restocking trip at the current location vs. the expected cost along the remainder of the route.

Stochastic VRP: An Overview and an Exact Method for Optimal Restocking
Solution approach

- We consider an exact solution approach based on the Integer L-Shaped method of Laporte and Louveaux.
- The algorithm is coded in C++.
- The branching routine is implemented by using the OOB B package coded at the CIRRELT.
- Subtour elimination and stochastic capacity constraints are separated using the CVRPSEP package of Lysgaard et al. (2004).
Solution approach (2)

- To enhance running times, we use different families of lower bounding functionals (valid inequalities) pertaining to partial routes, as in Jabali et al. (2014).

- The lower bounding functionals derive from the idea originally proposed by Hjorring and Holt (1999)

- Generalizing these lower bounding functionals to deal with the new recourse policies is definitely non-trivial!!!
An Exact Algorithm to Solve the Vehicle Routing Problem with Stochastic Demands under an Optimal Restocking Policy

M. Gendreau(1), M. Salavati(2), O. Jabali(3), W. Rei(4)

(1) : CIRRELT and MAGI, École Polytechnique de Montréal
(2) : CIRRELT and DIRO, Université de Montréal
(3) : CIRRELT and DEIB, Politecnico di Milano
(4) : CIRRELT and ESG, Université du Québec à Montréal
Recourse Policy: Traditional Recourse

Traditional Recourse Policy

Follow the planned route.
Execute a **BF trip** whenever route failures observed.

\[ \xi_j > q \]
Recourse Policy: Traditional Recourse

Traditional Recourse Policy

Follow the planned route. Execute a BF trip whenever route failures observed.

$$Q + \xi_j - q$$

![Diagram of a network with nodes labeled $v_1, v_2, v_3, v_{j-1}, v_{j+1}, v_{j+2}, v_t, v_{t-1}$ and a depot labeled DEPOT. The diagram shows a network with edges connecting the nodes, and trucks are depicted at some of the nodes.]
Recourse Policy: Traditional Recourse

Traditional Recourse Policy

Follow the planned route.
Execute a **BF trip** whenever route failures observed.
A Rule-Based Recourse for VRPSD

Rule-Based Recourse Policy

Follow the planned route. Execute **BF trips** when route failure occurred. Execute **PR trips** based on fixed thresholds.

\[ 0 \leq q - \xi_j < \theta ? \]
A Rule-Based Recourse for VRPSD

Rule-Based Recourse Policy

Follow the planned route. Execute BF trips when route failure occurred. Execute PR trips based on fixed thresholds.

\[ 0 \leq q - \xi_j < \theta^? = \delta Q \]
A Rule-Based Recourse for VRPSD

Rule-Based Recourse Policy

Follow the planned route. Execute BF trips when route failure occurred. Execute PR trips based on fixed thresholds.

\[ 0 \leq q - \xi_j < \theta ? = \eta E \xi_{j+1} \]
A Rule-Based Recourse for VRPSD

Rule-Based Recourse Policy

Follow the planned route. Execute BF trips when route failure occurred. Execute PR trips based on fixed thresholds.

\[
0 \leq q - \xi_j < \theta^? = \lambda \sum_{k=j+1}^{t} E \xi_k
\]
A Rule-Based Recourse for VRPSD

Rule-Based Recourse Policy

Follow the planned route. Execute BF trips when route failure occurred. Execute PR trips based on fixed thresholds.

\[ 0 \leq q - \xi_j < \theta^? = \lambda \sum_{k=j+1}^t E\xi_k \]
Recourse Policy:

Rule-Based Recourse Policy

Given planned route $i$ as $\tilde{v} = (v_1 = v_{i1}, \ldots, v_{ij}, v_{i,j+1}, \ldots, v_{it}, v_{i,t+1} = v_1)$.

Given threshold $\tilde{\theta} = (\theta_{i2}, \ldots, \theta_{it})$ for route $\tilde{v}$.

We define Recourse Function $F_{ij}(q)$ as the expected recourse cost starting from vertex $v_{ij}$ with $q$ units of residual capacity:

Recourse Function for Computing Expected Recourse Cost

$$F_{ij}(q) = \begin{cases} 
\sum_{s: \xi_{ij}^s > q} \left( b + 2c_{1ij} + F_{ij+1}(Q + q - \xi_{ij}^s) \right) p_{ij}^s + \\
\sum_{s: q - \theta_{ij} < \xi_{ij}^s \leq q} \left( c_{1ij} + c_{1ij+1} - c_{ij+1} + F_{ij+1}(Q) \right) p_{ij}^s + \\
\sum_{s: \xi_{ij}^s \leq q - \theta_{ij}} F_{ij+1}(Q - \xi_{ij}^s) p_{ij}^s & \text{if } j = 2, \ldots, t
\end{cases}$$

$$Q_{\tilde{v},1} = F_{i1}(Q), \quad Q(\tilde{v}) = \min\{Q^1_{\tilde{v}}, Q^2_{\tilde{v}}\}$$ (2)
Literature Review

Volume-based Recourse Policies:

1. Traditional Recourse proposed by Dror and Trudeau (1986).

Risk-and-Distance-based Recourse Policy:

Optimal Restocking Policy

How it works?

Compute proceeding cost-to-go and replenishment cost-to-go. Choose optimal action by choosing the action which incurs the min. cost.
Optimal Restocking Policy

How it works?

Compute **proceeding cost-to-go** and **replenishment cost-to-go**. Choose **optimal action** by choosing the action which incurs the **min. cost**.
Optimal Restocking Policy

How it works?

Compute proceeding cost-to-go and replenishment cost-to-go. Choose optimal action by choosing the action which incurs the min. cost.
Optimal Restocking Policy

How it works?

Compute **proceeding cost-to-go** and **replenishment cost-to-go**. Choose **optimal action** by choosing the action which incurs the **min. cost**.

Replenish
Optimal Restocking Policy

How it works?

Compute proceeding cost-to-go and replenishment cost-to-go. Choose optimal action by choosing the action which incurs the min. cost.
## Optimal Restocking Policy

### Recourse Policy: Optimal Policy

**Optimal threshold** $\theta_{ij}^*$

By comparing `proceed` and `replenish` we can compute optimal thresholds as follows:

$$\theta_{ij}^* = \min_{0, \ldots, Q} \{ \tilde{q} | \text{proceed} \leq \text{replenish} \}$$

Then:

- PR trip if $\tilde{q} < \theta_{ij}^*$
- proceed otherwise

**Function $F_{ij}(\tilde{q})$**

$$F_{ij}(\tilde{q}) = \min \begin{cases} 
\text{proceed}: c_{ij, i_{j+1}} + \sum_{k: \xi_{k, i_{j+1}} \leq \tilde{q}} F_{i_{j+1}}(\tilde{q} - \xi_{k, i_{j+1}}) p_{i_{j+1}}^k + \sum_{k: \xi_{k, i_{j+1}} > \tilde{q}} [b + 2c_{i_{j+1}} + F_{i_{j+1}}(Q + \tilde{q} - \xi_{k, i_{j+1}})] p_{i_{j+1}}^k, \\
\text{replenish}: c_{1, i_{j}} + c_{1, i_{j+1}} + \sum_{k=1}^{s} F_{i_{j+1}}(Q - \xi_{k, i_{j+1}}) p_{i_{j+1}}^k. 
\end{cases}$$
Exact Algorithms to Tackle VRPSD

**Integer L-shaped algorithm**

*Integer L-shaped algorithm* as a general *B&C* framework is proposed by Laporte and Louveaux (1993). Gendreau et al. (1995), Laporte et al. (2002), and Jabali et al. (2014)...

**Column Generation**


*B&C&P*: Gauvin et al. (2014).
Two-stage Stochastic Integer Program with Recourse

\[
\text{minimize } \sum_{i<j} c_{ij} x_{ij} + Q(x) \quad (3)
\]

subject to

\[
\sum_{j=2}^{n} x_{1j} = 2m, \quad (4)
\]

\[
\sum_{i<k} x_{ik} + \sum_{k<j} x_{kj} = 2, \quad k = 2, \ldots, n \quad (5)
\]

\[
\sum_{v_i, v_j \in S} x_{ij} \leq |S| - \left[ \frac{\sum_{v_i \in S} E(\xi_i)}{Q} \right], \quad (S \subset V \setminus \{v_1\}; 2 \leq |S| \leq n - 2) \quad (6)
\]

\[
0 \leq x_{ij} \leq 1, \quad 2 \leq i < j < n \quad (7)
\]

\[
0 \leq x_{1j} \leq 2, \quad j = 2, \ldots, n \quad (8)
\]

\[
x = (x_{ij}), \quad \text{integer} \quad (9)
\]

where, \( Q(x) = \sum_{k} \min\{Q^{k,1}, Q^{k,2}\} \).
## Solution Method: Integer L-shaped Alg. (General B&C)

### First Relaxation: Current Problem \((CP^0)\)

1. Replace \(Q(x)\) by \(\Theta\) in (1).
2. Bound \(\Theta\) from below by \(L\) in (11).
3. Relax subtour elimination and capacity constraints (4).
4. Relax integrality requirements (7).

### At Each Iteration

1. Solve relaxation.
2. Add subtour elimination and capacity constraints (4) when violated.
3. Add LBF cuts (10) to improve lower bounds on \(\Theta\).
4. Branching to achieve integrality requirements (7).
5. Add optimality cuts (11) when \(Q(x) > \Theta\).
Optimal Restocking Policy  

**Solution Method: Integer L-shaped Alg. (General B&C)**

\[ CP : \min_{x, \Theta} \sum_{i<j} c_{ij} x_{ij} + \Theta \]  \hspace{1cm} (10)

subject to

\[ \sum_{v_i, v_j \in S^k} x_{ij} \leq |S^k| - \left[ \frac{\sum_{v_i \in S^k} E(\xi_i)}{Q} \right] \quad 2 \leq |S^k| \leq n - 2, \] \hspace{1cm} (11)

\[ L + (\Theta^q_p - L) \left( \sum_{h \in \text{PR}^q} W_p^h(x) - |\text{PR}^q| + 1 \right) \leq \Theta \quad p \in \{\alpha, \beta, \gamma\}, \] \hspace{1cm} (12)

\[ L \leq \Theta \] \hspace{1cm} (13)

\[ \sum_{1 \leq i \leq j} x_{ij} \leq \sum_{1 \leq i \leq j} x_{ij}^f - 1 \] \hspace{1cm} (14)
Integer L-Shaped Algorithm

1. List contains initial relaxed CP $\bar{Z} := EEV$
2. Choose next pendent node
3. $Z^* \geq \bar{Z}$
   - yes: fathom node
   - no: Introduce violated feasibility constraints and LBFs
4. apply branching
5. Integer
   - yes: Add optimality cuts
     - yes: update $\bar{Z}$
     - no: no
   - no: no
6. Introduce violated feasibility constraints and LBFs
An Exact Algorithm

Feasibility cuts
Remove infeasible routes by adding capacity and subtour elimination constraints (CVRP package Lysgaard et al 2014).

Bounding Scheme for Fractional Solutions
Add valid inequalities that improve lower bound at fractional solutions with certain structures
This requires to compute expected recourse cost of such fractional solutions.

General Lower Bound
Compute a general lower bound $L$ that improves optimality gap.
Literature Review

Bounding Scheme for Fractional Solutions

- Hjorring and Holt (1999) introduce the concept of partial route for single-VRPSD (traditional, discrete, restricted failures)
- Laporte et al. (2002) generalize for multi-VRPSD (traditional, continuous)
- Jabali et al. (2014) generalize the structure of partial routes (traditional, continuous)
An Exact Algorithm

Bounding Scheme for Fractional Solutions

Add valid inequalities that improve lower bound at fractional solutions with certain structures. This requires to compute expected recourse cost of such fractional solutions.

Fractional Solution

Fractional Solutions with certain structures are called partial routes.
An Exact Algorithm

Fractional Solution

Partial Routes proposed by Jabali et al. (2014).

\[ v_1 \]
\[ h_1 \]
\[ v_1 \]
\[ h_1 \]
\[ v_2 \]
\[ h_1 \]
\[ v_2 \]
\[ h_1 \]
\[ \alpha \text{ topology} \]
\[ \beta \text{ topology} \]
\[ \gamma \text{ topology} \]
An Exact Algorithm

Bounding Scheme for Fractional Solutions

Add valid inequalities that improve lower bound at fractional solutions with certain structures.
This requires to compute expected recourse cost of such fractional solutions.

Expected Recourse Cost

First: we model a partial route with \( \alpha \) topology as an artificial route.

\[ h = (DEPOT = v_{i_1}, \ldots, v_{i_{j-l}}, i_{j-l+1}, \ldots, i_k, i_{k+1}, \ldots, i_j, v_{i_{j+l}}, \ldots, v_{i_{t+1}} = DEPOT) \]
An Exact Algorithm: An Optimal Policy for VRPSD

Bounding Scheme for Fractional Solutions

This requires to compute expected recourse cost of such fractional solutions.

Expected Recourse Cost

Conditional optimal cost-to-go from \( i_a \) when \( i_{a+1} \) is restricted to \( v_{u_1} \in U_{h}^{1} \) can be computed as \( \hat{F}_{i_a}(s = (i_a, q), s' = (v_{u_1}, q')) \):

\[
\hat{F}_{i_a}(s = (i_a, q), s' = (v_{u_1}, q')) = \min \left\{ \sum_{k : \xi_{u_1}^k \leq q} \tilde{F}_{i_{a+1}}(s' = (v_{u_1}, q' := q - \xi_{u_1}^k))p_{u_1}^k + \sum_{k : \xi_{u_1}^k > q} [b + 2c_{1,u_1} + \tilde{F}_{i_{a+1}}(s' = (v_{u_1}, q' := Q + q - \xi_{u_1}^k))]p_{u_1}^k, c_{1,i_a} + c_{1,u_1} - c_{i_k,u_1} + \sum_{k=1}^{s_{u_1}} \tilde{F}_{i_{a+1}}(s' = (v_{u_1}, q' := Q - \xi_{u_1}^k))p_{u_1}^k \right\}
\]

\[
\tilde{F}_{i_a}(s = (i_a, q)) = \min_{v_{u_e} \in U_{h}^{1}} \hat{F}_{i_a}(s = (i_a, q), s' = (v_{u_e}, q')). \tag{15}
\]
An Exact Algorithm

General Lower Bound

Compute a general lower bound $L$ that improves optimality gap

Definition

The general lower bound $L$ is defined by Laporte and Louveaux (1993) as a feasible integer solution (i.e., $m$ vehicle routes) with the least expected recourse cost defined as follows

$$L = Q(x^L) \leq \min_x \{ Q(x) \mid x \text{ is feasible} \}$$
An Exact Algorithm

General Lower Bound

Compute a general lower bound $L$ that improves optimality gap

Computation

We define a Mega partial route including all customers except one, then

$$L^* = \min_{S^2_h} Q(Mega) \leq L = Q(x^L)$$

using bounding scheme.
An Exact Algorithm

General Lower Bound

Compute a general lower bound $L$ that improves optimality gap

How we can use a Mega structure to compute $L$?

Define \textit{Mega} as

$$\tilde{l}_z = (v_1 = v_{i_1}, \ldots, v_{i_2}, \ldots, v_{i_{t-1}}, v_z, v_{i_{t+1}} = v_1).$$

for each vertex $v_z$.

Finally, a general lower bound $L^*$ can be computed as

$$L^* = \min_{z:2,\ldots,n} \tilde{F}_{v_1}(Q) - \sum_{k=1}^{m-1} c_{PR}^k.$$

where, $c_{PR}^k$ denotes the $k^{th}$ least PR trip cost ($1 \leq k \leq m$) and prove that $L^* \leq L$, so $L^*$ is a general valid lower bound for Expected Recourse cost.
Three Groups of Instances

1. Symmetric Instances.
2. Asymmetric Instances.
Instance Generation

Symmetric Instances

Three demand ranges $[1, 5]$, $[6, 10]$, and $[11, 15]$ are considered, and customers are assigned randomly to them. Probabilities $\{0.1, 0.2, 0.4, 0.2, 0.1\}$ are associated accordingly. 
Number of runs $= 11 \times 10 \times 4 = 440$.

Asymmetric Instances

Five demand ranges $[1, 5]$, $[6, 10]$, $[11, 15]$, $[4, 7]$, and $[9, 12]$ are considered, and customers are assigned randomly to them. Probabilities $\{0.1, 0.2, 0.4, 0.2, 0.1\}$ are associated accordingly for first three ranges and $\{0.4, 0.3, 0.2, 0.1\}$ for last two. 
Number of runs $= 11 \times 10 \times 4 = 440$. 
Numerical Results

Symmetric: VRP Performance Measures

Optimally solved inst., gap, and running time

Table: Result of running the optimal policy.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>( \bar{f} )</th>
<th>Opt</th>
<th>( \leq 1% )</th>
<th>Run(sec)</th>
<th>Gap</th>
<th>( \bar{f} )</th>
<th>Opt</th>
<th>( \leq 1% )</th>
<th>Run(sec)</th>
<th>Gap</th>
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## Optimally solved inst., gap, and running time

**Table:** Result of running the optimal restocking policy.

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<th>$\bar{f}$</th>
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<th>Run (sec)</th>
<th>Gap</th>
<th>$\bar{f}$</th>
<th>Opt</th>
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### Numerical Results

#### Asymmetric: VRP Performance Measures

Optimally solved inst., gap, and running time

**Table:** Result of running the optimal policy.

<table>
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<th>n</th>
<th>m</th>
<th>$\bar{f}$</th>
<th>Opt</th>
<th>≤ 1%</th>
<th>Run(sec)</th>
<th>Gap</th>
<th>$\bar{f}$</th>
<th>Opt</th>
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<th>Run(sec)</th>
<th>Gap</th>
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<td>− − −</td>
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Optimally solved inst., gap, and running time

Table: Result of running the optimal restocking policy.

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<tr>
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<th>m</th>
<th>(\bar{f})</th>
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<th>Run(sec)</th>
<th>Gap</th>
<th>(\bar{f})</th>
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Overall Performance

Table: Performance of Optimal Restocking Policy.

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<tr>
<th>Integer L-Shaped Algorithm</th>
<th>opt. sol.</th>
<th>time</th>
<th>gap</th>
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<tr>
<td></td>
<td>51.59%</td>
<td>1407.42</td>
<td>0.90%</td>
</tr>
</tbody>
</table>

Gendreau, Salavati, Jabali, and Rei CIRRELT (Centre interuniversitaire de recherche sur les réseaux d’entreprise, la logistique et le transport (CIRRELT), Montréal, Canada, MAGI, Polytechnique Montréal, Canada, DIRO, Université de Montréal, Canada, DEIB, Politecnico di Milano, Italy, ESG, Université du Québec à Montréal, Canada)
Savings: Sav1(savings on total cost) and Sav2(savings on recourse cost)

Table: Optimal restocking policy vs classical

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>( \bar{f} )</th>
<th>Sav1</th>
<th>Sav2</th>
<th>( \bar{f} )</th>
<th>Sav1</th>
<th>Sav2</th>
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<tbody>
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<td>0.42%</td>
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<td>0.92</td>
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</tr>
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<td>0.33%</td>
<td>31.21%</td>
<td>0.92</td>
<td>0.26%</td>
<td>40.78%</td>
</tr>
<tr>
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<td>2</td>
<td>0.90</td>
<td>0.08%</td>
<td>43.11%</td>
<td>0.92</td>
<td>0.31%</td>
<td>47.34%</td>
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<td>0.92</td>
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<td>4</td>
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</table>
**Savings: Sav1(savings on total cost) and Sav2(savings on recourse cost)**

**Table: Optimal restocking policy vs classical**

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>(\bar{f})</th>
<th>Sav1</th>
<th>Sav2</th>
<th>(\bar{f})</th>
<th>Sav1</th>
<th>Sav2</th>
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Savings: Sav1(savings on total cost) and Sav2(savings on recourse cost)

**Table**: Optimal restocking policy vs classical

<table>
<thead>
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<th>$n$</th>
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<th>$\bar{f}$</th>
<th>Sav1</th>
<th>Sav2</th>
<th>$\bar{f}$</th>
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<th>Sav2</th>
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Savings: Sav1 (savings on total cost) and Sav2 (savings on recourse cost)

Table: Optimal restocking policy vs classical

<table>
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<th>$n$</th>
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<th>Sav1</th>
<th>Sav2</th>
<th>$\bar{f}$</th>
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<th>Sav2</th>
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<td>48.42%</td>
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Louveaux and Salazar-Gonzalez (2017)

Instance Generation

Identical demands:
3 realizations: $\xi \in \{4, 5, 6\}$, $Pr : \{0.25, 0.5, 0.25\}$
9 realizations: $\xi \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$Pr : \{0.04, 0.08, 0.12, 0.16, 0.2, 0.16, 0.12, 0.08, 0.04\}$

\[
F_j(\tilde{q}) = \min \left\{ \begin{array}{l}
\text{Proceed : } \sum_{k : \xi_{j+1}^k \leq \tilde{q}} F_{j+1}(\tilde{q} - \xi_{j+1}^k)p_{j+1}^k + \\
\sum_{k : \xi_{j+1}^k > \tilde{q}} [\Delta + 2c_{0,j+1} + F_{j+1}(Q + \tilde{q} - \xi_{j+1}^k)]p_{j+1}^k,
\end{array} \right.
\]

\[
\text{Replenish : } \Delta + c_{0,j} + c_{0,j+1} - c_{j,j+1} + \sum_{k=1}^{s} F_{j+1}(Q - \xi_{j+1}^k)p_{j+1}^k.
\]
### Comparison with Louveaux and Salazar-Gonzalez (2017)

**Table: Result for $\Delta = 0$**

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## Comparison with Louveaux and Salazar-Gonzalez (2017)

**Table: Result for $\Delta = 0$**

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Comparison with Louveaux and Salazar-Gonzalez (2017)

Table: Result for $\Delta = 100$

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Numerical Results

Comparison with Louveaux and Salazar-Gonzalez (2017)

Table: Result for $\Delta = 100$

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Gendreau, Salavati, Jabali, and Rei CIRRELT (1 Centre interuniversitaire de recherche sur les réseaux d’entreprise, la logistique et le transport (CIRRELT), Montréal, Canada, 2 Centre interuniversitaire de recherche sur les logistiques et les transports (CIRRELT), Montréal, Canada, 3 DIRO, Université de Montréal, Canada, 4 DEIB, Politecnico di Milano, Italy, 5 ESG, Université du Québec à Montréal, Canada)
Conclusions and Perspectives
Conclusion and perspectives

- Stochastic vehicle routing is a rich and promising research area.
- Much work remains to be done in the area of recourse definition.
- SVRP models and solution techniques may also be useful for tackling problems that are not really stochastic, but which exhibit similar structures.
- Up to now, very little work on problems with stochastic travel and service times, while one may argue that travel or service times are uncertain in most routing problems!
- Correlation between uncertain parameters is possibly a major stumbling block in many application areas, but almost no one seems to work on ways to deal with it.